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2007 J. Phys. A: Math. Theor. 40 4665

(http://iopscience.iop.org/1751-8121/40/17/N01)

J. Phys. A: Math. Theor. **40** (2007) 4665–4669 doi:10.1088/1751-8113/40/17/N01

COMMENT

Comment on 'On the inconsistency of the Bohm–Gadella theory with quantum mechanics'

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Received 22 January 2007
Published 11 April 2007
Online at stacks.iop.org/JPhysA/40/4665

Abstract
Rigged Hilbert spaces of Hardy functions lead to a consistent theory of resonance scattering and decay. Contrary to the claims of a recent article [8], the theory holds for a wide range of potentials and rigorously describes the asymmetric time evolution of resonances.

PACS numbers: 03.65.−w, 02.30.Hq

1. Introduction

A recently published work [8] carries out an analysis of time asymmetric quantum theory (TAQT) of resonance scattering and decay. This theory is based on a pair of rigged Hilbert spaces (RHSs) of Hardy functions. As seen from the complete and systematic presentations, for instance [3–5], TAQT is based on a pair of rigged Hilbert spaces,

$$\Phi_\pm \subset \mathcal{H} \subset \Phi_\pm^\times. \tag{1.1}$$

By spectral theorem, $\Phi_\pm$ of (1.1) can be defined as vector spaces whose L^2-realizations are furnished by smooth, Hardy class functions, $\Phi_\pm := \mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}$. So defined, the spaces Φ_- and Φ_+ have the physical interpretation as the spaces of scattering in-states ϕ^+ and out-observables ψ^-, respectively. Under this construction, the heuristically introduced Gamow vectors $|z_R^-\rangle = \left|E - \mathrm{i}\frac{\Gamma}{2}^-\right\rangle$ and eigenkets $|E^\mp\rangle$ with incoming and outgoing boundary conditions can be well defined as elements of the duals $\Phi_+^\times$ and $\Phi_\pm^\times$, respectively.

The author of [8] does not dispute the mathematics of TAQT, in particular the construction of the RHSs (1.1) or their important properties such as semigroup time evolution and generalized eigenvectors of the Hamiltonian like $|z_R^-\rangle$ with complex eigenvalues. Rather, the main claim of [8] is that rigged Hilbert spaces of Hardy functions are 'inconsistent' with certain requirements of the physical theory such as the semiboundedness of the Hamiltonian. In particular, it has also been asserted in [8] that there are no potentials to which TAQT applies. Indeed, since $\left|E_R - \mathrm{i}\frac{\Gamma}{2}\right\rangle$ and $|E^\pm\rangle$ were originally introduced in connection with a suitably specified potential V such that $H = H_0 + V$, it is desirable that the wavefunctions

$\langle r|E^{\pm}\rangle$, where $|E^{\pm}\rangle \in \Phi^{\times}_{\mp}$, be solutions of the time-independent Schrödinger equation for some reasonable potentials.

However, note that the construction of the RHSs (1.1) holds for *any* potential V that satisfies the following two conditions:

(1) The full Hamiltonian $H = H_0 + V$ is essentially self-adjoint and has an absolutely continuous spectrum bounded from below.
(2) The Møller wave operators exist and are asymptotically complete.

Therefore, the assertion of [8] *that* 'nobody has found a potential to which such a theory applies' *is absurd*. In fact, among the potentials to which the theory applies is the very example of the spherical shell potential considered in [8]. In the next section, we will outline how RHSs of Hardy functions may be constructed for this potential using the techniques of [8] itself.

2. Spherical shell potential

Most of the conclusions of [8] about TAQT are drawn from the single example of scattering off a spherical shell potential. We will show here that this potential is perfectly consistent with TAQT and that the conclusions of [8] are the consequences of mathematical errors.

The analysis of [8] begins with the radial equation (for the $l = 0$ states)

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}r^2} + V(r)\right)\langle r|E^{\pm}\rangle = E\langle r|E^{\pm}\rangle \tag{2.1}$$

where $r \geqslant 0$, $V(r) = 1$ for $a < r < b$ and $V(r) = 0$ otherwise. The analyticity properties of $\chi^{\pm}(r; E) := \langle r|E^{\pm}\rangle$ are mentioned in [9]. The notation $\langle r|E^{\pm}\rangle$ suggests that $|r\rangle$ and $|E^{\pm}\rangle$ are functionals on some suitably defined vector spaces (e.g., test function spaces) and that $\langle r|E^{\pm}\rangle$ is an integral kernel. Thus, with the choice $\Phi := \mathcal{S}(\mathbb{R}^+/\{a, b\})$, i.e., Schwartz functions on $\mathbb{R}^+$ that vanish at the points a and b, the kets $|E^{\pm}\rangle$ are defined as antilinear functionals by the formula

$$\varphi^{\pm}(E) := \langle E^{\pm}|\varphi^{\pm}\rangle = \int_0^{\infty} \mathrm{d}r\, \varphi^{\pm}(r)\langle E^{\pm}|r\rangle = \int_0^{\infty} \mathrm{d}r\, \varphi^{\pm}(r)\overline{\langle r|E^{\pm}\rangle}. \tag{2.2}$$

It is claimed that $\varphi^{\pm}$ have complex analytic extensions which in turn follow from those of $|E^{\pm}\rangle$, e.g., $\varphi^{+}(z) = \int_0^{\infty} \mathrm{d}r\, \varphi^{+}(r)\langle z^{+}|r\rangle$ [8]. It is also claimed in [8] that the functions $\varphi^{\pm}(E)$, while admitting analytic extensions, cannot be of Hardy class. The author comes to this conclusion by considering the behaviour of functions $\varphi^{\pm}(z)$ for $z \to \infty$. To that end, a result attributed to [10] is used to obtain the inequality ((3.11) of [8])

$$|\chi^{-}(r; z)| := \left|\langle z^{+}|r\rangle\right| \leqslant C\frac{|z|^{1/4}r}{1 + |z|^{1/2}r}\mathrm{e}^{|\mathrm{Im}\sqrt{z}|r}. \tag{2.3}$$

From this, it has been concluded in [8] that 'when $z \in \mathbb{C}^{-}_{II}$, $\chi^{-}(r, z)$ blows up exponentially as z tends to infinity'.

The author then uses the bound (2.3) to obtain the behaviour of $\varphi^{+}(z)$ for $z \to \infty$. In particular, for $\varphi^{+}(r) \in \mathcal{C}^{\infty}$ with exponential falloff $\mathrm{e}^{-\frac{r^a}{a}}$, the author argues that $\varphi^{+}(z)$ blows up as $\mathrm{e}^{\frac{|\mathrm{Im}(\sqrt{z})|^b}{b}}$. For $\varphi^{+}(r) \in \mathcal{C}_0^{\infty}$, the author uses the inequality (2.3) to obtain

$$|\varphi^{+}(z)| \leqslant C\frac{|z|^{1/4}A}{1 + |z|^{1/2}A}\mathrm{e}^{|\mathrm{Im}\sqrt{z}|A} \tag{2.4}$$

and concludes 'when $\varphi^{+}(r) \in C_0^{\infty}$, $\varphi^{+}(z)$ blows up *exponentially* in the infinite arc of $\mathbb{C}^{-}_{II}$'.

These conclusions are simply wrong for nothing can be inferred about the behaviour of a function $\varphi^{+}(z)$ for $|z| \to \infty$ from the fact that it is bounded in the modulus by an exponential *like in* (2.3). Even if (2.3) and (2.4) are correct, all one can conclude is that $\chi^-(r;z)$ and $\varphi^+(z)$ are bounded by functions that blow up for $|z| \to \infty$, an obviously empty conclusion. The author makes the same mistake again in section 5 of [8] (inequality (5.11)). *Since the assertion that functions* $\varphi^\pm(z)$ *cannot be of Hardy class is solely based on this mistake, the central argument of* [8] *falls apart.*

On the other hand, even if boundary conditions for $\varphi^\pm(r)$ can be properly chosen so that energy wavefunctions $\varphi^\pm(E)$ are not of Hardy class, such a choice does not certainly exclude Hardy functions as possible solutions. In other words, even if it had been conducted with proper mathematical care, the study of [8] could not have led to a conclusion about the veracity of TAQT. In fact, for the potential used in [8], it is possible to choose a domain in which the Hamiltonian is essentially self-adjoint such that there exist RHSs of Hardy functions for the energy wavefunctions. To that end, consider the Hilbert spaces $L^2(\mathbb{R}^+, \mathrm{d}r)$ and $L^2(\mathbb{R}^+, \mathrm{d}E)$. Both are the same Hilbert space of square-integrable functions on the positive semiaxis, although we use a different notation for each because the former is the space of radial wavefunctions and the latter contains the same states in the energy representation. The mappings $U_\pm : L^2(\mathbb{R}^+, \mathrm{d}r) \longmapsto L^2(\mathbb{R}^+, \mathrm{d}E)$ defined by (2.2) are unitary. The inverse operators $U_\pm^{-1}$ are given by $\left(U_\pm^{-1}\widehat{f}^\pm\right)(r) = f(r) = \int_0^\infty \widehat{f}^\pm(E)\overline{\langle r|E^\pm\rangle}\,\mathrm{d}E$.

Now, let $\widehat{\varphi}^\pm \in \mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}$. Since $\widehat{\varphi}^\pm \in L^2(\mathbb{R}^+, \mathrm{d}E)$, we can use them in the integrand of the above integral defining $U_\pm^{-1}$ to obtain $\varphi^\pm = U_\pm^{-1}\widehat{\varphi}^+ \in L^2(\mathbb{R}^+, \mathrm{d}r)$. Since $U_\pm$ are unitary, the action of $U_\pm$ on $\varphi^\pm$ is to return the original functions $\widehat{\varphi}^\pm \in \mathcal{S} \cap \mathcal{H}^2_-\big|_{\mathbb{R}^+}$. Since $\mathcal{S}\cap\mathcal{H}^2_\pm\big|_{\mathbb{R}^+}$ are dense subspaces in which the Hamiltonian (defined as the multiplication operator) is essentially self-adjoint, by unitarity of $U_\pm^{-1}$ the collections of functions $\{\varphi^\pm\} = U_\pm^{-1}\left(\mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}\right)$ are dense subspaces of $L^2(\mathbb{R}^+, \mathrm{d}r)$ in which the Hamiltonian is essentially self-adjoint.

What this argument shows is that the same procedure used in [8] for constructing energy wavefunctions can be used to construct RHSs of Hardy functions; we only need to choose $U_\pm^{-1}\left[\mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}\right]$ as the domains of the Hamiltonian in the position representation. In this sense, albeit inadvertently, an explicit method of constructing the RHSs (1.1) of Hardy functions for the spherical shell potential has been given in [8]. The method runs closely parallel to the general procedure for constructing RHSs of Hardy functions [3]. The only remaining problem here is to identify the form and properties of those functions in $L^2(\mathbb{R}^+, \mathrm{d}r)$ that are in one-to-one correspondence with the subspaces $\mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}$ through $U_\pm$. While this requires further investigations, the main features of the rigged Hilbert spaces of Hardy functions have already been exemplified for the spherical shell potential.

3. Remarks on the literature cited in [8]

The article [8] also draws erroneous conclusions on the literature on TAQT. For instance, the authors of [2] do not take 'as a mathematical statement for 'no preparations for $t > 0$' the condition $|\langle E|\varphi^{\mathrm{in}}(t)\rangle| = |\langle^+E|\varphi^+(t)\rangle| = 0$ for $t > 0$', as the author of [8] paraphrases ((4.3) of [8]). Instead, in the original article [2], almost the same words are used to motivate a different mathematical statement ((3.4) of [2]): 'As the mathematical statement for 'no preparations for $t > 0$' we therefore write (the slightly weaker condition) $0 = \int \mathrm{d}E\,\langle E|\phi^{\mathrm{in}}(t)\rangle = \int \mathrm{d}E\,\langle^+E|\phi^+(t)\rangle = \int \mathrm{d}E\,\langle^+E|\mathrm{e}^{-\mathrm{i}tH}|\phi^+\rangle$'.

That is, the authors of [2] never use the 'assumption' (4.3) of [8] anywhere in their derivation. They do not use the vanishing of the integrand to conclude that integral (3.4) of [2] vanishes, as the author of [8] claims. It is (3.4) of [2] (or (4.4) of [8]) that the authors take as the mathematical statement for 'no preparations for $t \geqslant 0$' and not (4.3) of [8]. While it is certain that the assumption $\langle E, \eta|\phi^{\mathrm{in}}(t)\rangle = 0$ for $t > 0$ (or $\langle E^+, \eta|\phi^+(t)\rangle = 0$ for $t > 0$)

implies $\phi^{\text{in}} = 0$ (or $\phi^+ = 0$), it is no less certain that (3.4) of [2] does not imply (4.10) of [8]. According to the Paley–Wiener theorems [7], the mathematical formulation of 'no preparations for $t \geqslant 0$' given by (3.4) of [2] is equivalent to the hypothesis that $\langle E|\phi^{\text{in}}\rangle = \langle^+E|\phi^+\rangle$ is a Hardy function from below. Thus, the author has incorrectly quoted from [2] to arrive at the non-sensical result ((4.10) of [8]).

It is also inaccurate to state that the proponents of TAQT dispense with asymptotic completeness. What TAQT establishes is that the 'in' and 'out' scattering vectors inhabit two different vector spaces Φ_- and Φ_+, each of which is dense in the same Hilbert space. Therefore, the Hilbert space completion of Φ_+ and Φ_- gives $\mathcal{H}_{\text{in}} = \mathcal{H}_{\text{out}}$, the conventional principle of asymptotic completeness, and TAQT is not in contradiction with conditions that define the Møller operators, which play a role in the construction of (1.1), or the S-operator. However, TAQT does introduce a topological refinement to the principle of asymptotic completeness in that $\Phi_+ \neq \Phi_-$.

Furthermore, the spectrum of the Hamiltonian defined as the multiplication operator on the domains $\mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}$ is the half line $\mathbb{R}^+$ and not $\mathbb{R}$. The author of [8] may be confusing the fact that the functions of $\mathcal{S} \cap \mathcal{H}^2_\pm\big|_{\mathbb{R}^+}$ have extensions to $\mathbb{R}$ with the semiboundedness of the spectrum of H.

Finally, no questions exist about the construction of Gamow vectors by solving the Schrödinger equation with purely outgoing boundary conditions. There is a well-known procedure to obtain Gamow vectors this way [1, 5].

4. Concluding remarks

TAQT is an internally consistent, rigorous mathematical theory that describes both non-relativistic and relativistic resonance scattering and decay. The theory does not take the (norm complete) Hilbert space, therewith also the unitary time evolution, as fundamental but uses instead rigged Hilbert spaces of Hardy functions, leading to asymmetric, semigroup time evolutions. As such, we agree with the sentiment expressed by the title of [8]: if by 'standard quantum mechanics' the author means Hilbert spaces and unitary evolutions, indeed TAQT is not consistent with it. Aside from this, all the conclusions of [8] about TAQT are false. In fact, even if one can properly obtain solutions to the Schrödinger equation for the spherical shell potential that are not of Hardy class, this is not equivalent to saying that there are no Hardy class solutions for this potential or, much less, all other. Therefore, even if it had been conducted with proper mathematical care, the study of [8] could not have led to a conclusion about the veracity TAQT.

Acknowledgments

MG acknowledges financial support from the Junta de Castilla y León Project VA013C05 and Ministry of Education and Science of Spain, projects MTM2005-09183 and FIS2005-03988. SW acknowledges support from the University of Valladolid where he was a visitor while this work was done.

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